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On modified Weyl–Heisenberg algebras, noncommutativity, matrix-valued Planck constant and QM in Clifford spaces

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Abstract

A *novel* Weyl–Heisenberg algebra in Clifford spaces is constructed that is based on a matrix-valued $\mathcal{H}^{AB}$ extension of Planck's constant. As a result of this modified Weyl–Heisenberg algebra one will no longer be able to measure, simultaneously, the pairs of variables $(x, p_x), (x, p_y), (x, p_z), (y, p_x), \ldots$ with absolute precision. New Klein–Gordon and Dirac wave equations and dispersion relations in Clifford spaces are presented. The latter Dirac equation is a generalization of the Dirac–Lanczos–Barut–Hestenes equation. We display the explicit isomorphism between Yang's noncommutative spacetime algebra and the area-coordinates algebra associated with Clifford spaces. The former Yang's algebra involves noncommuting coordinates and momenta with a *minimum* Planck scale λ (ultraviolet cutoff) and a minimum momentum $p = \hbar/R$ (maximal length R, infrared cutoff). The double-scaling limit of Yang's algebra $\lambda \to 0, R \to \infty$, in conjunction with the large $n \to \infty$ limit, leads naturally to the *area quantization* condition $\lambda R = L^2 = n\lambda^2$ (in Planck area units) given in terms of the discrete angular-momentum eigenvalues n. It is shown how modified Newtonian dynamics is also a consequence of Yang's algebra resulting from the modified Poisson brackets. Finally, another noncommutative algebra which differs from Yang's algebra and related to the minimal length uncertainty relations is presented . We conclude with a discussion of the implications of noncommutative **QM** and **QFT**'s in Clifford spaces.

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1. Introduction

In recent years we have argued that the underlying fundamental physical principle behind string theory, not unlike the principle of equivalence and general covariance in Einstein's general relativity, might well be related to the existence of an invariant minimal length scale (Planck scale) attainable in nature. A scale relativistic theory involving spacetime *resolutions* was developed long ago by Nottale where the Planck scale was postulated as the minimum observer-independent invariant resolution [1] in nature. Since 'points' cannot be observed physically with an ultimate resolution, they are fuzzy and smeared out into fuzzy balls of Planck radius of arbitrary dimension. For this reason one must construct a theory that includes all dimensions (and signatures) on the equal footing. Because the notion of dimension is a topological invariant, and the concept of a fixed dimension is lost due to the fuzzy nature of points, dimensions are resolution dependent, one must also include a theory with *all* topologies as well. It is our belief that this may lead to the proper formulation of string and M theory.

In [2], we applied this extended scale relativity principle to the quantum mechanics of p-branes which led to the construction of C-space (a Clifford space, a dimension *category*) where all p-branes were taken to be on the same footing, i.e. transformations in C-space reshuffled a string history for a five-brane history, a membrane history for a string history, for example. It turned out that Clifford algebras contained the appropriate algebro-geometric features to implement this principle of poly-dimensional transformations [3–5].

Clifford algebras have been a very useful tool for a description of geometry and physics [4–8]. In [3, 5], it was proposed that every physical quantity is in fact a *poly-vector*, that is, a Clifford number or a Clifford aggregate. Also, spinors are the members of left- or right-minimal ideals of Clifford algebra, which may provide the framework for a deeper understanding of sypersymmetries, i.e., the transformations relating bosons and fermions. The Fock–Stueckelberg theory of a relativistic particle [4] can be embedded in the Clifford algebra of spacetime [3]. Many important aspects of Clifford algebra are described in [3, 5–8]. In particular, spinor representations in any dimensions and signatures in terms of projection operators and 'families' of spinors based on Clifford algebra objects were developed by [58] that might shed some light in understanding the families of quarks and leptons.

Using these methods the bosonic p-brane propagator, in the quenched mini superspace approximation, was constructed in [9]; the logarithmic corrections to the black hole entropy based on the geometry of Clifford space (in short C-space) were obtained in [14]; the action for higher derivative gravity with torsion from the geometry of C-spaces and how the Conformal algebra of spacetime emerges from the Clifford algebra was described in [29]; the resolution of the ordering ambiguities of QFT in curved spaces was resolved by [3].

In this new physical theory the arena for physics is no longer the ordinary spacetime, but a more general manifold of Clifford algebra-valued objects, poly-vectors. Such a manifold has been called a pan-dimensional continuum [5] or C-space [2]. The latter describes on a unified basis the objects of various dimensionality: not only points, but also closed lines, surfaces, volumes , . . . , called 0-loops (points), 1-loops (closed strings) 2-loops (closed membranes), 3-loops, etc. It is a sort of a *dimension* category, where the role of functorial maps is played by C-space transformations which reshuffles a p-brane history for a p'-brane history or a mixture of all of them, for example.

The above geometric objects may be considered as to corresponding to the well-known physical objects, namely closed p-branes. Technically, those transformations in C-space that reshuffle objects of different dimensions are generalizations of the ordinary Lorentz transformations to C-space. In that sense, the C-space is roughly speaking a sort of generalized Penrose twistor space from which the ordinary spacetime is a *derived* concept. In [2], we derived the minimal length uncertainty relations as well as the full blown uncertainty relations due to the contributions of *all* branes of *every* dimensionality, ranging from $p = 0$ all the way to $p = \infty$. In [14], we extended this derivation to include the *maximum* Planck temperature condition.

The contents of this work is the following. In sections 2.1 and 2.2, we will review the basic features of the extended relativity theory in C-spaces and the explicit derivation from first principles of the minimal length modified Heisenberg uncertainty relations. This derivation is based on the effective-running Planck 'constant' $\hbar_{\text{eff}}(p^2)$ (energy dependent) that results from a breakdown of poly-dimensional covariance in C-spaces.

In section 3.1, we show the relationship among Yang's 4D noncommutative spacetime algebra [17] (in terms of the 8D phase space coordinates), the *area coordinates* algebra of the C-space associated with a 6D Clifford algebra and the *Euclideanized* AdS_5 spaces. The role of AdS_5 was instrumental in explaining the origins of an extra (infrared) scale R in conjunction to the (ultraviolet) Planck scale λ characteristic of C-spaces. Tanaka [18] gave the physical and mathematical derivation of the *discrete* spectra for the spatial coordinates and spatial momenta that yields a minimum length-scale λ (ultraviolet cutoff in energy) and a minimum momentum $p = \hbar/R$ (maximal length R, infrared cutoff).

In section 3.2, the double-scaling limit, $\lambda \to 0$, $R \to \infty$, in conjunction with the large $n \to \infty$ limit, leads to the area-quantization condition $\lambda R = L^2 = n\lambda^2$ in units of the Planck area, where n is the angular momentum $\Sigma^{56} = (1/\hbar)\mathcal{M}^{56}$ eigenvalue. In general, the norm-squared of the area operator has a correspondence with the quadratic Casimir $\Sigma_{AB}\Sigma^{AB}$ of the conformal algebra $SO(4,2)(SO(5,1)$ in the Euclideanized AdS_5 case). This quadratic Casimir must not be confused with the $SU(2)$ Casimir J^2 with eigenvalues $j(j+1)$. It is shown how modified Newtonian dynamics is also a consequence of Yang's algebra resulting from the modified Poisson brackets.

In section 4, we proceed with the construction of the *modified* Weyl–Heisenberg algebra in C-spaces that is based on a matrix-valued $\mathcal{H}^{AB}$ Planck constant. As a result of the modified Weyl–Heisenberg algebra one will no longer be able to measure simultaneously the pairs of variables $(x, p_x), (x, p_y), (x, p_z), (y, p_x), \ldots$ with absolute precision. Novel **QM** Klein–Gordon, Dirac wave equations and dispersion relations in C-spaces are presented. In section 5, we discuss briefly another algebras associated with *noncommuting* poly-coordinates and poly-momenta that differ from the generalized Yang's noncommutative algebra in C-spaces displayed in [53]. In the final part of section 5, we analyse some of the future implications of **QM** and **QFT**'s in C-spaces.

2. The extended relativity in Clifford spaces

2.1. Extending relativity from Minkowski spacetime to C-space

We embark into the extended relativity theory in C-spaces by a natural generalization of the notion of a spacetime interval in Minkwoski space to C-space:

$$\mathrm{d}X^2 = \mathrm{d}\Omega^2 + \mathrm{d}X_\mu\,\mathrm{d}X^\mu + \mathrm{d}x_{\mu\nu}\,\mathrm{d}X^{\mu\nu} + \cdots. \tag{2.1}$$

The Clifford-valued poly-vector

$$X = X^M E_M = \Omega\mathbf{1} + x^\mu\gamma_\mu + x^{\mu\nu}\gamma_\mu\wedge\gamma_\nu + \cdots x^{\mu_1\mu_2\cdots\mu_D}\gamma_{\mu_1}\wedge\gamma_{\mu_2}\cdots\wedge\gamma_{\mu_D}. \tag{2.2a}$$

denotes the position of a polyparticle in a manifold, called Clifford space or C-space. The series of terms in (2) terminates at a *finite* value depending on the dimension D. A Clifford algebra $Cl(r,q)$ with $r+q = D$ has 2^D basis elements. For simplicity, the gammas γ^μ correspond to a Clifford algebra associated with a flat spacetime:

$$1/2\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}, \tag{2.2b}$$

but in general one could extend this formulation to curved spacetimes with metric $g^{\mu\nu}$.

The connection to strings and p-branes can be seen as follows. In the case of a closed string (a 1-loop) embedded in a target flat spacetime background of D-dimensions, one represents the 'holographic' projections [9, 10] of the closed string (1-loop) onto the embedding spacetime coordinate planes by the variables $x_{\mu\nu}$. These variables represent the respective *areas* enclosed by the 'holographic' projections of the closed string (1-loop) onto the corresponding embedding spacetime planes [9, 10]. Similarly, one can embed a closed membrane (a 2-loop) onto a D-dimensional flat spacetime, where the projections given by the antisymmetric variables $x_{\mu\nu\rho}$ represent the corresponding *volumes* enclosed by the projections of the 2-loop along the hyper-planes of the flat target spacetime background.

This procedure can be carried to all closed p-branes (p-loops) where the values of p are $p = 0, 1, 2, 3, \ldots, D-2$. The $p = 0$ value represents the centre of mass and the coordinates $x^{\mu\nu}, x^{\mu\nu\rho}, \ldots$ have been *coined* in the string-brane literature [9, 10] as the *holographic* areas, volumes, ... projections of the nested family of p-loops (closed p-branes) onto the embedding spacetime coordinate planes/hyper-planes.

The classification of Clifford algebras $Cl(r, q)$ in $D = r + q$ dimensions (modulo 8) for different values of the spacetime signature r, q is discussed, for example, in the book of Porteous [27]. All Clifford algebras can be understood in terms of $CL(8)$ and $CL(k)$ for k less than 8 due to the modulo 8 Periodicity theorem:

$$CL(n) = CL(8) \otimes Cl(n-8).$$

$Cl(r, q)$ is a matrix algebra for even $n = r + q$ or the sum of two matrix algebras for odd $n = r + q$. Depending on the signature, the matrix algebras may be real, complex, or quaternionic. For further details we refer to [27] . If we take the differential $\mathrm{d}x$ and compute the scalar product among two poly-vectors $\langle \mathrm{d}X^{\dagger}\,\mathrm{d}X\rangle_{\text{scalar}}$, we obtain the C-space extension of the particles proper time in Minkwoski space. The symbol X^{+} denotes the *reversion* operation and involves reversing the order of all the basis γ^{μ} elements in the expansion of X. It is the analogue of the transpose (Hermitian) conjugation. The C-space proper time associated with a polyparticle motion is then

$$\mathrm{d}\Sigma^2 = (\mathrm{d}\Omega)^2 + \Lambda^{2D-2}\,\mathrm{d}X_{\mu}\,\mathrm{d}X^{\mu} + \Lambda^{2D-4}\,\mathrm{d}x_{\mu\nu}\,\mathrm{d}X^{\mu\nu} + \cdots. \tag{2.3}$$

Here we have explicitly introduced the Planck scale Λ since a length parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops, ..., p-loops. Einstein introduced the speed of light as a universal absolute invariant in order to 'unite' space with time (to match units) in the Minkwoski space interval:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \mathrm{d}x_i\,\mathrm{d}x^i. \tag{2.4}$$

A similar unification is needed here to 'unite' objects of different dimensions, such as x^{μ}, $x^{\mu\nu}$, etc. The Planck scale then emerges as another universal invariant in constructing an extended scale relativity theory in C-spaces [2]. To continue along the same path, we consider the analogue of Lorentz transformations in C-spaces which transform a poly-vector X into another poly-vector X' given by $X' = RXR^{-1}$ with

$$R = \mathrm{e}^{\theta^A E_A} = \exp[(\theta\mathbf{1} + \theta^{\mu}\gamma_{\mu} + \theta^{\mu_1\mu_2}\gamma_{\mu_1}\wedge\gamma_{\mu_2}\cdots)] \tag{2.5}$$

and

$$R^{-1} = \mathrm{e}^{-\theta^A E_A} = \exp[-(\theta\mathbf{1} + \theta^{\nu}\gamma_{\nu} + \theta^{\nu_1\nu_2}\gamma_{\nu_1}\wedge\gamma_{\nu_2}\cdots)], \tag{2.6}$$

where the theta parameters in (2.5) and (2.6) are the components of the Clifford-valued parameter $\Theta = \theta^A E_A$:

$$\theta, \theta^{\mu}, \theta^{\mu\nu}, \ldots. \tag{2.7}$$

They are the C-space version of the Lorentz rotations/boosts parameters.

Since a Clifford algebra admits a matrix representation, one can write the norm of a poly-vectors in terms of the trace operation as $\|X\|^2 = \text{Trace } X^2$. Hence under C-space Lorentz transformation the norms of poly-vectors behave like as follows:

$$\text{Trace } X'^2 = \text{Trace}[RX^2R^{-1}] = \text{Trace}[RR^{-1}X^2] = \text{Trace } X^2. \qquad (2.8a)$$

These norms are invariant under C-space Lorentz transformations due to the cyclic property of the trace operation and $RR^{-1} = 1$.

Another way of rewriting the inner product of poly-vectors is by means of the *reversal* operation $\sim$, not to be confused with the Hermitian operation $\dagger$, that reverses the order of the Clifford basis generators: $(\gamma^\mu \wedge \gamma^\nu)^\sim = \gamma^\nu \wedge \gamma^\mu$, etc. Hence the inner product can be rewritten as the scalar part of the geometric product $\langle X^\sim X\rangle_s$. The analogue of an orthogonal matrix in Clifford spaces is $R^\sim = R^{-1}$ such that

$$\langle X'^\sim X'\rangle_s = \langle (R^{-1})^\sim X^\sim R^\sim RXR^{-1}\rangle_s = \langle RX^\sim XR^{-1}\rangle_s = \langle X^\sim X\rangle_s = \text{invariant}. \qquad (2.8b)$$

This condition $R^\sim = R^{-1}$, of course, will *restrict* the type of terms allowed inside the exponential defining the rotor R in equations (2)–(5) because the *reversal* of a p-vector obeys

$$(\gamma_{\mu_1} \wedge \gamma_{\mu_2} \cdots \wedge \gamma_{\mu_p})^\sim = \gamma_{\mu_p} \wedge \gamma_{\mu_{p-1}} \cdots \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_1} = (-1)^{p(p-1)/2}\gamma_{\mu_1} \wedge \gamma_{\mu_2} \cdots \wedge \gamma_{\mu_p}. \qquad (2.9)$$

Hence only those terms that *change* sign (under the reversal operation) are permitted in the exponential defining $R = \exp[\theta^A E_A]$. For example, in $D = 4$, in order to satisfy the condition $R^\sim = R^{-1}$, one must have from the behaviour under the reversal operation expressed in equations (2)–(9) that

$$R = \exp\left[\theta^{\mu_1\mu_2}\gamma_{\mu_1} \wedge \gamma_{\mu_2} + \theta^{\mu_1\mu_2\mu_3}\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}\right]. \qquad (2.10a)$$

such that

$$\begin{aligned} R^\sim &= \exp\left[\theta^{\mu_1\mu_2}(\gamma_{\mu_1} \wedge \gamma_{\mu_2})^\sim + \theta^{\mu_1\mu_2\mu_3}(\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3})^\sim\right] \\ &= \exp\left[-\theta^{\mu_1\mu_2}\gamma_{\mu_1} \wedge \gamma_{\mu_2} - \theta^{\mu_1\mu_2\mu_3}\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}\right] = R^{-1}. \end{aligned} \qquad (2.10b)$$

These transformations are the analogue of Lorentz transformations in C-spaces which transform a poly-vector X into another poly-vector X' given by $X' = RXR^{-1}$. The theta parameters $\theta^{\mu_1\mu_2}, \theta^{\mu_1\mu_2\mu_3}$ are the C-space version of the Lorentz rotations/boosts parameters. The ordinary Lorentz rotation/boosts involves only the $\theta^{\mu_1\mu_2}\gamma_{\mu_1} \wedge \gamma_{\mu_2}$ terms, because the Lorentz algebra generator can be represented as $\mathcal{M}^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$. The $\theta^{\mu_1\mu_2\mu_3}\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \gamma_{\mu_3}$ are the C-space corrections to the ordinary Lorentz transformations when $D = 4$.

Another possibility is to *complexify* the C-space poly-vector-valued coordinates = $Z = Z^A E_A = X^A E_A + \mathrm{i}Y^A E_A$ and the boosts/rotation parameters θ allowing the unitarity condition $U^\dagger = U^{-1}$ to hold in the generalized Clifford unitary transformations $Z' = UZU^\dagger$ associated with the complexified poly-vector $Z = Z^A E_A$ such that the interval

$$\langle \mathrm{d}Z^\dagger\, \mathrm{d}Z\rangle_s = \mathrm{d}\bar\Omega\, \mathrm{d}\Omega + \mathrm{d}\bar z^\mu\, \mathrm{d}z_\mu + \mathrm{d}\bar z^{\mu\nu}\, \mathrm{d}z_{\mu\nu} + \mathrm{d}\bar z^{\mu\nu\rho}\, \mathrm{d}z_{\mu\nu\rho} + \cdots \qquad (2.11)$$

remains invariant (upon setting the Planck scale $\Lambda = 1$).

The unitary condition $U^\dagger = U^{-1}$ under the *combined* reversal and complex-conjugate operation will constrain the form of the complexified boosts/rotation parameters θ^A appearing in the rotor: $U = \exp[\theta^A E_A]$. The theta parameters θ^A are either purely real or purely imaginary depending if ${E_A}^\dagger = \pm E_A$, to ensure that an overall *change* of sign occurs in the terms $\theta^A E_A$ inside the exponential defining U so that $U^\dagger = U^{-1}$ holds and the norm $\langle Z^\dagger Z\rangle_s$ remains invariant under the analogue of unitary transformations in *complexified* C-spaces.

These techniques are not very different from Penrose twistor spaces. As far as we know a Clifford twistor space construction of C-spaces has not been performed so far.

Another alternative is to define the polyrotations by $R = \exp(\Theta^{AB}[E_A, E_B])$, where the commutator $[E_A, E_B] = f_{ABC}E_C$ is the C-space analogue of the $\mathrm{i}[\gamma_\mu, \gamma_\nu]$ commutator which is the generator of the Lorentz algebra, and the theta parameters Θ^{AB} are the C-space analogues of the rotation/boots parameters $\theta^{\mu\nu}$. The diverse parameters Θ^{AB} are purely real or purely imaginary depending whether the reversal $[E_A, E_B]^\sim = \pm[E_A, E_B]$ to ensure that $R^\sim = R^{-1}$ so that the scalar part $\langle X^\sim X\rangle_s$ remains invariant under the transformations $X' = RXR^{-1}$. This last alternative seems to be more physical because a poly-rotation should map the E_A direction into the E_B direction in C-spaces, hence the meaning of the generator $[E_A, E_B]$ which extends the notion of the $[\gamma_\mu, \gamma_\nu]$ Lorentz generator.

Another immediate application of this theory is that one may consider 'strings' and 'branes' in C-spaces as a unifying description of *all* branes of different dimensionality. In fact, a unified action of *all* p-branes was written in [44]. As we have already indicated, since spinors are left/right ideals elements of a Clifford algebra, a supersymmetry is then naturally incorporated into this approach as well. In particular, one can have world volume and target space supersymmetry *simultaneously* [44]. A generalized poly-vector-valued supersymmetry based on Clifford spaces was attained in [54], and extensions of the standard model based on generalized tensorial gauge theories can also be found in [54]. We hope that the C-space 'strings' and 'branes' may lead us towards discovering the physical foundations of string and M-theory. For other alternatives to supersymmetry see the work by Chisholm and Baylis [33]. A flat C-space does not mean it has a trivial geometry. A flat C-space has nontrivial *torsion*. The analogue of the scalar curvature in C-spaces can be decomposed as sums of powers of the Riemann curvature scalar (and other contractions of the curvature tensors) including torsion terms [13]. Thus, relativity in C-spaces involves a higher derivative gravity theory with *torsion* [44], and a vanishing cosmological constant in C-spaces does not amount to a vanishing cosmological constant in ordinary spacetimes [44] .

Related to the minimal Planck scale, an upper limit on the maximal acceleration principle in nature was proposed by Cainello [28]. This idea is a direct consequence of a suggestion made years earlier by Max Born on a dual relativity principle operating in phase spaces [32]. There is an upper bound on the four-force (maximal string tension or tidal forces in the string case) acting on a particle as well as an upper bound in the particles velocity. One can combine the maximum speed of light with a minimum Planck scale into a maximal proper acceleration $a = c^2/\Lambda$ within the framework of Finsler geometry [16]. A thorough study of Finsler geometry and Clifford algebras has been undertaken by Vacaru [35] . Other several new physical implications of the maximal acceleration principle in nature, like neutrino oscillations, have been studied by [34]. A variable fine structure constant, with the cosmological expansion of the Universe, has been proposed by us where an exact renormalization group-like equation governing the cosmological time (scale) variation of the fine structure constant was derived explicitly from this maximal acceleration principle in nature [37].

2.2. The generalized String/Brane uncertainty relations

Below we will review how the minimal length string uncertainty relations can be obtained from the polyparticle dynamics C-spaces [2]. The truly C-space invariant norm of a momentum poly-vector is defined (after introducing suitable powers of the Planck scale in the sum in order to match units):

$$\|P\|^2 = \pi^2 + p_\mu p^\mu + p_{\mu\nu}p^{\mu\nu} + p_{\mu\nu\rho}p^{\mu\nu\rho} + \cdots = M^2. \tag{2.12a}$$

A detailed discussion of the physical properties of all the components of the polymomentum P in four dimensions and the emergence of the physical mass m in Minkowski spacetime has been provided in the book by Pavsic [3]. The polymomentum in $D = 4$ can be written as

$$P = P^A E_A = \mu + p^\mu \gamma_\mu + S^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \pi^\mu \gamma_5 \gamma_\mu + m \gamma_5, \tag{2.12b}$$

where the pseudo-scalar component $m\gamma_5$ is the one which contains the physical mass in Minkwoski spacetime. This justifies using the notation m for mass.

The most salient feature of the polyparticle dynamics in C-spaces is that one can start with a *constrained* action in C-space and arrive, nevertheless, at an *unconstrained* Stuckelberg action in Minkowski space (a subspace of C-space). It follows that p_μ is a constant of motion $p_\mu p^\mu = m^2$ but m is no longer a *fixed* constant entering the action but it is now an *arbitrary* constant of motion. The true constraint in C-space is

$$\|P\|^2 = P_A P^A = \mu^2 + p_\mu p^\mu + \pi_\mu \pi^\mu - m^2 - 2S^{\mu\nu} S_{\mu\nu} = M^2. \tag{2.12c}$$

This is basically the distinction between the variable m and the fixed constant M. The variable m is the conjugate to the Stuckelberg evolution parameter s that allowed Pavsic to propose a natural solution of the problem of time in quantum cosmology [3]. Equation (2.12c) is a generalization (more degrees of freedom) of the de Sitter top studied in [48].

Nottale has given convincing arguments why the notion of *dimension* is resolution dependent, and at the Planck scale, the minimum attainable distance, the dimension becomes singular, that is blows-up. Setting aside at this moment the potential algebraic convergence problems when $D = \infty$, if we take the dimension at the Planck scale to be infinity, then the norm P^2 will involve an infinite number of terms. It is precisely this infinite series expansion which will reproduce *all* the different forms of the Casimir invariant masses appearing in kappa-deformed Poincare algebras [11, 12]. As mentioned earlier, when $D = \infty$, the Planck scale appearing in the series expansion of (2.12) $\Lambda_\infty = G^{1/0} = 1$.

It was discussed recently why there is an infinity of possible values of the Casimirs invariant M^2 due to an infinite choice of possible bases. The parameter κ is taken to be equal to the inverse of the Planck scale. The classical Poincare algebra is retrieved when $\Lambda = 0$. The kappa-deformed Poincare algebra does *not* act in classical Minkwoski spacetime. It acts in a quantum-deformed spacetime. We conjecture that the natural deformation of Minkowski spacetime is given by C-space.

The way to generate different expressions for the M^2 is by taking slices (sections) of the 2^D-dimensional mass-shell hyper-surface in C-space onto subspaces of smaller dimensionality. This is achieved by imposing the following constraints on the holographic components of the poly-vector-momentum. In doing so, one is explicitly *breaking* the poly-dimensional covariance and for this reason one can obtain an infinity of possible choices for the Casimirs M^2.

To demonstrate this, we impose the following constraints:

$$p_{\mu\nu} p^{\mu\nu} = a_2 (p_\mu p^\mu)^2 = a_2 p^4 \qquad p_{\mu\nu\rho} p^{\mu\nu\rho} = a_3 (p_\mu p^\mu)^3 = a_3 p^6 . \cdots . \tag{2.13}$$

What the terms of equation (2.13) represent *physically* is the *breaking* of the full poly-dimensional covariance in C-space down to a direct sum of $SO(N)$ subgroups given by $SO(D(D-1)/2) \oplus SO(D(D-1)(D-2)/3) \oplus \cdots$. Equation (2.13) represents geometrically the slicing of the 2^D-dimensional mass-shell hyper-surface in C-space into $(D(D-1)/2)$-, $(D(D-1)(D-2)/3)$-, . . . -dimensional hyper-spherical regions (subspaces). For example when $D = 4$, the first term of equation (2.13) represents the six-dimensional spherical region of radius $\sqrt{a_2 p^4}$ which is parametrized by the six coordinates p^{01}, p^{02}, p^{03}, p^{12}, p^{13}, p^{23}. The second term of equation (2.13) represents the four-dimensional spherical region of radius

$\sqrt{a_3 p^6}$ which is parametrized by the four coordinates p^{012}, p^{013}, p^{023}, p^{123}, etc. The radii of those $(D(D-1)/2)$-, $(D(D-1)(D-2)/3)$-, ... -dimensional hyper-spherical regions (subspaces) are parametrized solely in terms of the ordinary momentum coordinates p_μ and the parameters a_n. This is the reason why we decided to choose such constraint (2.13). There are many different ways to perform the slicing procedure in C-space depending on the choices of the coefficients a_n, i.e, there are many ways to break the full poly-dimensional covariance in C-space. Upon doing so the norm of the poly-momentum becomes

$$\|P\|^2 = P_A P^A = \sum_{n=0}^{n=D} a_n p^{2n} = M^2(a_o, a_2, a_3, \ldots, a_D). \tag{2.14}$$

Therefore, by a judicious choice of the coefficients a_n, and by reinserting the suitable powers of the Planck scale, which have to be there in order to combine objects of different dimensions, one can reproduce *all* the possible Casimirs in the form

$$M^2 = m^2[f(\Lambda m/\hbar)]^2 \qquad m^2 \equiv p_\mu p^\mu = p^2. \tag{2.15}$$

To illustrate the relevance of poly-vectors, we will summarize our derivation of the minimal length string uncertainty relations [2]. Because of the existence of the extra holographic variables $x^{\mu\nu}, \ldots$ one cannot naively impose $[x, p] = \mathrm{i}\hbar$ due to the effects of the other components. The units of $[x_{\mu\nu}, p^{\mu\nu}]$ are of $\hbar^2$ and of higher powers of $\hbar$ for the other commutators. To achieve covariance in C-space which reshuffles objects of different dimensionality, the effective Planck constant in C-space should be given by a sum of powers of $\hbar$.

This is not surprising. Classical C-space contains the Planck scale, which itself depends on $\hbar$. This implies that already at the classical level, C-space contains the seeds of the quantum space. At the next level of quantization, we have an effective $\hbar$ that comprises all the powers of $\hbar$ induced by the commutators involving *all* the holographic variables. In general one must write down the commutation relations in terms of poly-vector-valued quantities. In particular, the Planck constant will now be a Clifford number, a poly-vector with multiple components. This will be the subject of section 4.

The simplest way to infer the effects of the holographic coordinates of C-space on the commutation relations is by working with the effective $\hbar$ that appears in the *nonlinear* de Broglie dispersion relation. The mass-shell condition in C-space, after imposing the constraints among the holographic components, yields an effective mass $M = mf(\Lambda m/\hbar)$. The generalized de Broglie relations, which are *no* longer linear, are [2]

$$\begin{gathered}
|P_{\text{effective}}| = |p| f(\Lambda m/\hbar) = \hbar_{\text{effective}}(k^2)|k| \\
\hbar_{\text{effective}}(k^2) = \hbar f(\Lambda m/\hbar) = \hbar \sum_{n=0}^{n=N} a_n(\Lambda m/\hbar)^{2n} = \hbar \sum_{n=0}^{n=N} a_n(\Lambda k)^{2n}, \\
m^2 = p^2 = p_\mu p^\mu = (\hbar k)^2,
\end{gathered} \tag{2.16}$$

where the upper limit in the sum $N = D$ is given by the spacetime dimension. Using the effective $\hbar_{\text{eff}}$ in the well-known relation based on the Schwartz inequality and the fact that $|z| \geqslant |\mathrm{Im} z|$ leads to

$$\Delta x^i \Delta p^j \geqslant \tfrac{1}{2}\|\langle [x^i, p^j]\rangle\| \qquad [x^i, p^j] = \mathrm{i}\hbar_{\text{eff}}(k^2)\delta^{ij}. \tag{2.17}$$

In Euclidean spacetime one has that the norms obey the condition

$$\hbar^2 k^2 = \|p_\mu p^\mu\| = m^2 = \|(p_0)^2 + (\vec{p})^2\| \geqslant \|(\vec{p})^2\|. \qquad m^2 \geqslant \|(\vec{p})^2\|. \tag{2.18}$$

By choosing a *positive* sign of the numerical coefficients $a_n > 0$ in equation (2.16) it yields

$$a_n m^{2n} \geqslant a_n \|(\vec{p})^2\|^n = a_n[(p_1)^2 + (p_2)^2 + \cdots (p_{D-1})^2]^n \quad \Rightarrow \quad a_n m^{2n} \geqslant a_n (p_1)^{2n}, \\ a_n m^{2n} \geqslant a_n (p_2)^{2n}, \qquad a_n m^{2n} \geqslant a_n (p_3)^{2n}, \ldots, a_n m^{2n} \geqslant a_n (p_n)^{2n} (\text{no sum}). \tag{2.19}$$

From the conditions of equation (2.19) one learns that equation (2.16) obeys the inequality

$$\hbar_{\text{effective}}(k^2) = \hbar f(\Lambda m/\hbar) = \hbar \sum_{n=0}^{n=N} a_n (\Lambda m/\hbar)^{2n} \geqslant \hbar \sum_{n=0}^{n=N} a_n (\Lambda/\hbar)^{2n} (p_n)^{2n}. \tag{2.20}$$

The use of the inequalities

$$\langle p^2 \rangle \geqslant (\Delta p)^2, \quad \langle p^4 \rangle \geqslant (\Delta p)^4, \ldots, \langle p^{2n} \rangle \geqslant (\Delta p)^{2n} \tag{2.21}$$

in equation (2.20) leads to

$$\langle \hbar_{\text{effective}}(k^2) \rangle \geqslant \hbar \sum_{n=0}^{n=N} a_n (\Lambda/\hbar)^{2n} (\langle (p_n)^{2n} \rangle) \geqslant \hbar \sum_{n=0}^{n=N} a_n (\Lambda/\hbar)^{2n} (\Delta p)^{2n}. \tag{2.22}$$

Finally, by recurring to the result of equation (2.22) in equation (2.17), we get that for each pair of conjugate canonical variables $(x, p_x), (y, p_y), (z, p_z), \ldots$ the product of uncertainties (we omit indices for simplicity) is given by

$$\Delta x \Delta p \geqslant \frac{1}{2}\hbar + \frac{a_1 \hbar}{2} \left(\frac{\Lambda}{\hbar}\right)^2 (\Delta p)^2 + \frac{a_2 \hbar}{2} \left(\frac{\Lambda}{\hbar}\right)^4 (\Delta p)^4 + \cdots. \tag{2.23}$$

The second term of the last relation yields the stringy contribution to the modified uncertainty relations , whereas the higher order corrections in equation (2.21) stem from the higher rank components of the poly-momentum and represent the membrane, 3-brane. . . and $(D-1)$-brane contributions to the generalized uncertainty relations given by

$$\Delta x \geqslant \frac{\hbar}{2\Delta p} + \frac{a_1}{2} \frac{\Lambda^2}{\hbar} \Delta p + \frac{a_2}{2} \frac{\Lambda^4}{\hbar^3} (\Delta p)^3 + \cdots. \tag{2.24}$$

By replacing lengths by times and momenta by energy one reproduces the minimal Planck time uncertainty relations. By keeping only the first two terms of equation (2.24) one can infer that there is a *minimum* uncertainty of the order of the Planck scale Λ.

The *physical* interpretation of these uncertainty relations follow from the extended relativity principle. As we boost the string to higher trans-Planckian energies, part of the energy will *always* be invested into the strings potential energy, increasing its length into bits of Planck scale sizes, so that the original string will decompose into two, three, four, . . . strings of Planck sizes carrying units of Planck momentum, i.e. the notion of *a* single-particle/string loses its meaning beyond that point. This reminds one to ordinary relativity, where boosting a massive particle to higher energies will increase the speed while part of the energy is also invested into increasing its mass. In this process the speed of light remains the maximum attainable speed (it takes an infinite energy to reach it) and in our scheme the Planck scale is never surpassed. The effects of a minimal length can be clearly seen in Finsler geometries [16] having both a maximum four acceleration c^2/Λ (maximum tidal forces) and a maximum speed . The Riemannian limit is reached when the maximum four acceleration goes to infinity, i.e. the Finsler geometry 'collapses' to a Riemannian one.

3. The noncommutative spacetime Yang's algebra and C-spaces

3.1. The area coordinates noncommutative algebra, Clifford and Yang's algebras

The main result of this section is that there is a *subalgebra* of the C-space operator-valued coordinates which is *isomorphic* to the noncommutative Yang's spacetime algebra [17]. This, in conjunction to the discrete spectrum of angular momentum, leads to the discrete area-quantization in multiples of Planck areas. Namely, the 4D Yang's noncommutative spacetime (YNST) algebra [17] (written in terms of 8D phase-space coordinates) is isomorphic to the 15-dimensional *subalgebra* of the C-space operator-valued coordinates associated with the *holographic areas* of C-space. This connection between Yang's algebra and the 6D Clifford algebra is possible because the 8D phase-space coordinates x^μ, p^μ (associated with a 4D spacetime) have a one-to-one correspondence to the $\hat{X}^{\mu 5}$; $\hat{X}^{\mu 6}$ holographic area-coordinates of the C-space (corresponding to the 6D Clifford algebra).

Furthermore, Tanaka [18] has shown that Yang's algebra [17] (with 15 generators) is related to the 4D conformal algebra (15 generators) which in turn is isomorphic to a subalgebra of the 4D Clifford algebra because it is known that the 15 generators of the 4D conformal algebra $SO(4, 2)$ can be explicitly realized in terms of the 4D Clifford algebra as [29]

$$P^\mu = \mathcal{M}^{\mu 5} + \mathcal{M}^{\mu 6} = \gamma^\mu(\mathbf{1} + \gamma^5) \qquad K^\mu = \mathcal{M}^{\mu 5} - \mathcal{M}^{\mu 6} = \gamma^\mu(\mathbf{1} - \gamma^5)$$
$$D = \gamma^5 \qquad M^{\mu\nu} = \mathrm{i}[\gamma^\mu, \gamma^\nu] \cdots, \tag{3.1}$$

where the Clifford algebra generators

$$\mathbf{1}, \quad \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma^5 \tag{3.2}$$

account for the extra *two* directions within the C-space associated with the 4D Clifford algebra leaving effectively $4 + 2 = 6$ degrees of freedom that match the degrees of freedom of a 6D spacetime [29]. The relevance of [29] is that it was not necessary to work directly in 6D to find a realization of the 4D conformal algebra $SO(4, 2)$. It was possible to attain this by recurring solely to the 4D Clifford algebra as shown in equation (3.1).

One can also view the 4D conformal algebra $SO(4, 2)$ realized in terms of a 15-dimensional *subalgebra* of the 6D Clifford algebra. The bi-vector holographic area-coordinates $X^{\mu\nu}$ couple to the basis generators $\Gamma_\mu \wedge \Gamma_\nu$. The bi-vector coordinates $X^{\mu 5}$ couple to the basis generators $\Gamma_\mu \wedge \Gamma_5$ where now the Γ^5 is another generator of the 6D Clifford algebra and *mustnot* be confused with the usual γ^5 defined by equation (3.2). The bi-vector coordinates $X^{\mu 6}$ couple to the basis generators $\Gamma_\mu \wedge \Gamma_6$. The bi-vector coordinate X^{56} couples to the basis generator $\Gamma_5 \wedge \Gamma_6$.

In view of this fact that these bi-vector holographic area-coordinates in 6D *couple* to the bi-vectors basis elements $\Gamma_\mu \wedge \Gamma_\nu, \ldots,$ and whose algebra is in turn isomorphic to the 4D conformal algebra $SO(4, 2)$ via the realization in terms of the 6D angular momentum generators (and boosts generators) $\mathcal{M}^{\mu\nu} \sim [\Gamma^\mu, \Gamma^\nu]$, $\mathcal{M}^{\mu 5} \sim [\Gamma^\mu, \Gamma^5], \ldots$ we shall *define* the holographic *area coordinates algebra* in C-space as the *dual* algebra to the $SO(4, 2)$ conformal algebra (realized in terms of the 6D angular momentum, boosts, generators in terms of a 6D Clifford algebra generators as shown)

Note that the conformal boosts K^μ and the translations P^μ in equation (3.1) do commute $[P^\mu, P^\nu] = [K^\mu, K^\nu] = 0$, and for this reason we shall assign the appropriate correspondence $p^\mu \leftrightarrow X^{\mu 6}$ and $x^\mu \leftrightarrow X^{\mu 5}$, up to numerical factors (lengths) to match dimensions, in order to attain *noncommuting* variables x^μ, p^μ .

Therefore, one has two possible routes to relate Yang's algebra with Clifford algebras. One can relate Yang's algebra with the holographic area-coordinates algebra in the C-space associated with a 6D Clifford algebra and/or to the subalgebra of a 4D Clifford algebra via the realization of the conformal algebra $SO(4, 2)$ in terms of the 4D Clifford algebra generators $\mathbf{1}, \gamma^5, \gamma^\mu$ as shown in equation (3.1).

Since the relation between the 4D conformal and Yang's algebra and the implications for the AdS/CFT, dS/CFT duality have been discussed before by Tanaka [18], in this work we shall establish the following correspondence between the C-space holographic-area coordinates algebra (associated with the 6D Clifford algebra) and Yang's spacetime algebra via the angular momentum generators in 6D as follows:

$$\mathrm{i}\hat{M}^{\mu\nu} = \mathrm{i}\hbar\Sigma^{\mu\nu} \leftrightarrow \mathrm{i}\frac{\hbar}{\lambda^2}\hat{X}^{\mu\nu}. \tag{3.3}$$

$$\mathrm{i}\hat{M}^{56} = \mathrm{i}\hbar\Sigma^{56} \leftrightarrow \mathrm{i}\frac{\hbar}{\lambda^2}\hat{X}^{56}. \tag{3.4}$$

$$\mathrm{i}\lambda^2\Sigma^{\mu5} = \mathrm{i}\lambda\hat{x}^{\mu} \leftrightarrow \mathrm{i}\hat{X}^{\mu5}. \tag{3.5}$$

$$\mathrm{i}\lambda^2\Sigma^{\mu6} = \mathrm{i}\lambda^2\frac{R}{\hbar}\hat{p}^{\mu} \leftrightarrow \mathrm{i}\hat{X}^{\mu6}. \tag{3.6}$$

With Hermitian (bi-vector) operator coordinates

$$(\hat{X}^{\mu\nu})^\dagger = \hat{X}^{\mu\nu} \qquad (\hat{X}^{\mu5})^\dagger = \hat{X}^{\mu5} \qquad (\hat{X}^{\mu6})^\dagger = \hat{X}^{\mu6} \qquad (\hat{X}^{56})^\dagger = \hat{X}^{56}. \tag{3.7}$$

The algebra generators can be realized as

$$\hat{X}^{\mu\nu} = \mathrm{i}\lambda^2\left(X^\mu\frac{\partial}{\partial X_\nu} - X^\nu\frac{\partial}{\partial X_\mu}\right). \tag{3.8a}$$

$$\hat{X}^{\mu5} = \mathrm{i}\lambda^2\left(X^\mu\frac{\partial}{\partial X_5} - X^5\frac{\partial}{\partial X_\mu}\right). \tag{3.8b}$$

$$\hat{X}^{\mu6} = \mathrm{i}\lambda^2\left(X^\mu\frac{\partial}{\partial X_6} - X^6\frac{\partial}{\partial X_\mu}\right). \tag{3.8c}$$

$$\hat{X}^{56} = \mathrm{i}\lambda^2\left(X^5\frac{\partial}{\partial X_6} - X^6\frac{\partial}{\partial X_5}\right), \tag{3.8d}$$

where the angular momentum generators are defined as usual:

$$\hat{M}^{\mu\nu} \equiv \hbar\Sigma^{\mu\nu} \qquad \hat{M}^{\mu5} \equiv \hbar\Sigma^{\mu5} \qquad \hat{M}^{\mu6} \equiv \hbar\Sigma^{\mu6} \qquad \hat{M}^{56} \equiv \hbar\Sigma^{56}, \tag{3.8e}$$

which have a one-to-one correspondence to the Yang noncommutative spacetime (YNST) algebra generators in 4D. These generators (angular momentum differential operators) act on the coordinates of a 5D hyperboloid AdS_5 space defined by

$$-(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 - (x^6)^2 = R^2, \tag{3.9a}$$

where R is the *throat* size of the hyperboloid. This introduces an extra and crucial scale in addition to the Planck scale. Note that $\eta^{55} = +1$, $\eta^{66} = -1$. 5D de Sitter space dS_5 has the topology of $S^4 \times R^1$. Whereas AdS_5 space has the topology of $R^4 \times S^1$ and its conformal (projective) boundary at infinity has a topology $S^3 \times S^1$. Whereas the *Euclideanized* Anti de Sitter space AdS_5 can be represented geometrically as two disconnected branches (sheets) of a 5D hyperboloid embedded in 6D. The topology of these two disconnected branches is that of a 5D disc and the metric is the Lobachevsky one of constant negative curvature. The conformal group $SO(4, 2)$ leaves the 4D lightcone at infinity invariant.

Thus, *Euclideanized* AdS_5 is defined by a Wick rotation of the x^6 coordinate giving

$$-(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 + (x^6)^2 = R^2, \tag{3.9b}$$

whereas de Sitter space dS_5 with the topology of a pseudo-sphere $S^4 \times R^1$ and *positive* constant scalar curvature is defined by

$$-(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 + (x^5)^2 + (x^6)^2 = -R^2. \tag{3.9c}$$

(Note that Tanaka [18] uses *different* conventions than ours in his definition of the 5D hyperboloids. He has a sign change from R^2 to $-R^2$ because he introduces i factors in iR).

After this discussion and upon a direct use of the correspondence in equations (3.3)–(3.6), . . . yields the exchange algebra between the position and momentum coordinates:

$$[\hat{X}^{\mu 6}, \hat{X}^{56}] = -\mathrm{i}\lambda^2\eta^{66}\hat{X}^{\mu 5} \leftrightarrow \left[\frac{\lambda^2 R}{\hbar}\hat{p}^\mu, \lambda^2\Sigma^{56}\right] = -\mathrm{i}\lambda^2\eta^{66}\lambda\hat{x}^\mu \tag{3.10}$$

from which we can deduce that

$$[\hat{p}^\mu, \Sigma^{56}] = -\mathrm{i}\eta^{66}\frac{\hbar}{\lambda R}\hat{x}^\mu, \tag{3.11}$$

and after using the definition $\mathcal{N} = (\lambda/R)\Sigma^{56}$ one has the exchange algebra commutator of p^μ and $\mathcal{N}$ of Yang's spacetime algebra:

$$[\hat{p}^\mu, \mathcal{N}] = -\mathrm{i}\eta^{66}\frac{\hbar}{R^2}\hat{x}^\mu. \tag{3.12}$$

The other commutator is

$$[\hat{X}^{\mu 5}, \hat{X}^{56}] = -[\hat{X}^{\mu 5}, \hat{X}^{65}] = \mathrm{i}\eta^{55}\lambda^2\hat{X}^{\mu 6} \leftrightarrow [\lambda\hat{x}^\mu, \lambda^2\Sigma^{56}] = \mathrm{i}\eta^{55}\lambda^2\lambda^2\frac{R}{\hbar}\hat{p}^\mu \tag{3.13}$$

from which we can deduce that

$$[\hat{x}^\mu, \Sigma^{56}] = \mathrm{i}\eta^{55}\frac{\lambda R}{\hbar}\hat{p}^\mu, \tag{3.14}$$

and after using the definition $\mathcal{N} = (\lambda/R)\Sigma^{56}$ one has the exchange algebra commutator of x^μ and $\mathcal{N}$ of Yang's spacetime algebra:

$$[\hat{x}^\mu, \mathcal{N}] = \mathrm{i}\eta^{55}\frac{\lambda^2}{\hbar}\hat{p}^\mu. \tag{3.15}$$

The other relevant holographic area-coordinates commutators in C-space are

$$[\hat{X}^{\mu 5}, \hat{X}^{\nu 5}] = -\mathrm{i}\eta^{55}\lambda^2\hat{X}^{\mu\nu} \leftrightarrow [\hat{x}^\mu, \hat{x}^\nu] = -\mathrm{i}\eta^{55}\lambda^2\Sigma^{\mu\nu}. \tag{3.16}$$

after using the representation of the C-space operator holographic area-coordinates

$$\mathrm{i}\hat{X}^{\mu\nu} \leftrightarrow \mathrm{i}\lambda^2\frac{1}{\hbar}\mathcal{M}^{\mu\nu} = \mathrm{i}\lambda^2\Sigma^{\mu\nu}\mathrm{i}\hat{X}^{56} \leftrightarrow \mathrm{i}\lambda^2\Sigma^{56}, \tag{3.17}$$

where we appropriately introduced the Planck scale λ as one should to match units.

From the correspondence

$$\hat{p}^\mu = \frac{\hbar}{R}\Sigma^{\mu 6} \leftrightarrow \frac{\hbar}{R}\frac{1}{\lambda^2}\hat{X}^{\mu 6}, \tag{3.18}$$

one can obtain nonvanishing momentum commutator

$$[\hat{X}^{\mu 6}, \hat{X}^{\nu 6}] = -\mathrm{i}\eta^{66}\lambda^2\hat{X}^{\mu\nu} \leftrightarrow [\hat{p}^\mu, \hat{p}^\nu] = -\mathrm{i}\eta^{66}\frac{\hbar^2}{R^2}\Sigma^{\mu\nu}. \tag{3.19}$$

The signatures for AdS_5 space are $\eta^{55} = +1$, $\eta^{66} = -1$ and for the *Euclideanized* AdS_5 space are $\eta^{55} = +1$ and $\eta^{66} = +1$. Yang's spacetime algebra corresponds to the latter case.

Finally, the *modified* Heisenberg algebra can be read from the following C-space commutators:

$$\begin{aligned} &[\hat{X}^{\mu 5}, \hat{X}^{\nu 6}] = \mathrm{i}\eta^{\mu\nu}\lambda^2\hat{X}^{56} \leftrightarrow \\ &[\hat{x}^\mu, \hat{p}^\mu] = \mathrm{i}\hbar\eta^{\mu\nu}\frac{\lambda}{R}\Sigma^{56} = \mathrm{i}\hbar\eta^{\mu\nu}\mathcal{N}. \end{aligned} \tag{3.20}$$

Equations (3.12), (3.15), (3.16), (3.19) and (3.20) are the defining relations of Yang's noncommutative 4D spacetime algebra involving the 8D phase-space variables. These commutators obey the Jacobi identities. There are other commutation relations such as $[\mathcal{M}^{\mu\nu}, x^\rho], [\mathcal{M}^{\mu\nu}, p^\rho]$ that we did not write down. These are just the well-known rotations (boosts) of the coordinates and momenta. An immediate consequence of Yang's noncommutative algebra is that now one has modified products of uncertainties

$$\Delta x^\mu \Delta p^\nu \geqslant \hbar \eta^{\mu\nu} |\langle \Sigma^{56} \rangle|; \qquad \Delta x^\mu \Delta x^\nu \geqslant \lambda^2 |\langle \Sigma^{\mu\nu} \rangle|;$$
$$\Delta p^\mu \Delta p^\nu \geqslant \left(\frac{\hbar}{R}\right)^2 |\langle \Sigma^{\mu\nu} \rangle|. \tag{3.21a}$$

A generalization of Yang's noncommutative spacetime algebra to the full Clifford space involving poly-coordinates and poly-momenta was attained in [53]. Since the poly-vector-valued coordinates and momenta do not commute, one has uncertainty relations of the form

$$\Delta x^{\mu_1\mu_2\cdots\mu_n} \Delta p^{\mu_1\mu_2\cdots\mu_n} \geqslant \hbar^n, \qquad \Delta x^{\mu_1\mu_2\cdots\mu_n} \Delta x^{\nu_1\nu_2\cdots\nu_n} \geqslant \lambda^{2n},$$
$$\Delta p^{\mu_1\mu_2\cdots\mu_n} \Delta p^{\nu_1\nu_2\cdots\nu_n} \geqslant \left(\frac{\hbar}{R}\right)^{2n}. \tag{3.21b}$$

There is *no* summation of indices on the lhs and we have *omitted* the numerical factors and indices stemming from the generalized Kronecker deltas and the structure functions appearing on the rhs of equation (3.21b). These generalized uncertainty relations and the *quantization* of areas, volumes, hyper-volumes in units of the Planck scale will be the subject of future investigation. Noncommutative p-branes actions based on a novel Moyal–Yang star product deformations of the Nambu–Poisson brackets with an upper and lower scale was provided in [55]. It was also shown how QM wave equations in a D-dimensional noncommutative Yang's spacetime could be obtained from ordinary QM wave equations based on spaces with commuting coordinates and momenta in *higher* dimensions $(D + 2)$. For details we refer to [55].

3.2. The double scaling limit, area quantization and modified Newtonian mechanics

In this section, we will discuss in detail the double scaling limit and the modified Poisson brackets leading to modified Newtonian dynamics and resulting from Yang's algebra. When $\lambda \to 0$ and $R \to \infty$, one recovers the ordinary *commutative* spacetime algebra. The Snyder algebra [22] is recovered by setting $R \to \infty$ while leaving λ intact. To recover the ordinary Weyl–Heisenberg algebra is more subtle. Tanaka [18] has shown the *spectrum* of the operator $\mathcal{N} = (\lambda/R)\Sigma^{56}$ is discrete given by $n(\lambda/R)$. This is not surprising since the angular momentum generator $\mathcal{M}^{56}$ associated with the *Euclideanized* AdS_5 space is a rotation in the now compact $x^5 - x^6$ directions. This is not the case in AdS_5 space since $\eta^{66} = -1$ and this timelike direction is no longer compact. Rotations involving timelike directions are equivalent to noncompact boosts with a continuous spectrum.

In order to recover the standard Weyl–Heisenberg algebra from Yang's noncommutative spacetime algebra, and the standard uncertainty relations $\Delta x \Delta p \geqslant \hbar$ with the ordinary $\hbar$ term, rather than the $n\hbar$ term, one needs to take the limit $n \to \infty$ limit in such a way that the net combination of $n\frac{\lambda}{R} \to 1$.

This can be attained when one takes the *double* scaling limit of the quantities as follows:

$$\lambda \to 0 \qquad R \to \infty \qquad \lambda R \to L^2.$$
$$\lim_{n\to\infty} n\frac{\lambda}{R} = n\frac{\lambda^2}{\lambda R} = \frac{n\lambda^2}{L^2} \to 1. \tag{3.22a}$$

From equation (3.21) one learns then that

$$n\lambda^2 = \lambda R = L^2. \tag{3.22b}$$

The spectrum n corresponds to the quantization of the angular momentum operator in the $x^5 - x^6$ direction (after embedding the 5D hyperboloid of throat size R onto 6D). Tanaka [18] has shown why there is a *discrete spectra* for the *spatial* coordinates and *spatial* momenta in Yang's spacetime algebra that yields a *minimum* length λ (ultraviolet cutoff in energy) and a minimum momentum $p = \hbar/R$(maximal length R, infrared cutoff). The energy and temporal coordinates had a continuous spectrum.

The physical interpretation of the double-scaling limit of equation (3.22) is that the area $L^2 = \lambda R$ becomes now *quantized* in units of the Planck area λ^2 as $L^2 = n\lambda^2$. Thus the quantization of the area (via the double scaling limit) $L^2 = \lambda R = n\lambda^2$ is a result of the *discrete* angular momentum spectrum in the $x^5 - x^6$ directions of Yang's noncommutative spacetime algebra when it is realized by (angular momentum) differential operators acting on the *Euclideanized* AdS_5 space (two branches of a 5D hyperboloid embedded in 6D). A general interplay between quantum of areas and quantum of angular momentum, for arbitrary values of spin, in terms of the square root of the Casimir $\mathbf{A} \sim \lambda^2\sqrt{j(j+1)}$, has been obtained a while ago in loop quantum gravity by using spin-networks techniques and highly technical area-operator regularization procedures [41]. The advantage of this work is that we have arrived at similar (not identical) area-quantization conclusions in terms of minimal Planck areas and a discrete angular momentum spectrum n via the double scaling limit based on Clifford algebraic methods (C-space holographic area-coordinates). This is not surprising since the norm-squared of the holographic area operator has a correspondence with the quadratic Casimir $\Sigma_{AB}\Sigma^{AB}$ of the conformal algebra $SO(4,2)$ ($SO(5,1)$ in the Euclideanized AdS_5 case). This quadratic Casimir must not be confused with the $SU(2)$ Casimir J^2 with eigenvalues $j(j+1)$. Hence, the correspondence given by equations (3.3)–(3.8) gives $\mathbf{A}^2 \leftrightarrow \lambda^4\Sigma_{AB}\Sigma^{AB}$.

In [46], we have shown why AdS_4 gravity with a topological term, i.e. an Einstein–Hilbert action with a cosmological constant plus Gauss–Bonnet terms can be obtained from the vacuum state of a BF-Chern–Simons–Higgs theory *without* introducing by *hand* the zero torsion condition imposed in the MacDowell–Mansouri–Chamsedine–West construction. One of the most salient features of [46] was that a *geometric mean* relationship was derived among the cosmological constant Λ_c, the Planck area λ^2 and the AdS_4 throat size squared R^2 given by $(\Lambda_c)^{-1} = (\lambda)^2(R^2)$. Upon setting the throat size to coincide with the Hubble scale R_H one obtains the observed value of the vacuum energy density $\Lambda_c = L_{\text{Planck}}^{-2}R_H^{-2} = L_P^{-4}(L_P/R_H)^2 \sim 10^{-120}(M_{\text{Planck}})^4$. A similar geometric mean relation is also obeyed by the condition $\lambda R = L^2 (= n\lambda^2)$ in the double scaling limit of Yang's algebra which suggests to identify the cosmological constant as $\Lambda_c = L^{-4}$. This geometric mean condition remains to be investigated further. In particular, we presented the preliminary steps how to construct a noncommutative gravity via the Vasiliev–Moyal star products deformations of the $SO(4,2)$ algebra used in the study of higher conformal massless spin theories in AdS spaces by taking the inverse-throat size $1/R$ as a deformation parameter of the $SO(4,2)$ algebra [46]. A Moyal deformation of ordinary gravity via $SU(\infty)$ gauge theories was advanced in [31]. A new realization of holography and the geometrical interpretation of AdS_{2n} spaces in terms of $SO(2n-1,2)$ instantons was studied in [45].

Since the expectation value

$$\frac{\lambda^2}{L^2}\langle n|\Sigma^{56}|n\rangle = \frac{n\lambda^2}{L^2} = 1 \tag{3.23}$$

in the double-scaling limit, one recovers the standard Heisenberg uncertainty relations

$$\Delta x^\mu \Delta p^\mu \geqslant \tfrac{1}{2}||\langle[x^\mu, p^\mu]\rangle|| = \tfrac{1}{2}\hbar \tag{3.24}$$

and the commutators become in the double-scaling limit:

$$[\hat{p}^\mu, \Sigma^{56}] = -\mathrm{i}\eta^{66}\frac{\hbar}{L^2}\hat{x}^\mu \qquad [\hat{p}^\mu, \mathcal{N}] = 0 \tag{3.25}$$

$$[\hat{x}^\mu, \Sigma^{56}] = -\mathrm{i}\eta^{55}\frac{L^2}{\hbar}\hat{p}^\mu \qquad [\hat{x}^\mu, \mathcal{N}] = 0 \tag{3.26}$$

$$[\hat{x}^\mu, \hat{x}^\nu] = [\hat{p}^\mu, \hat{p}^\nu] = 0 \qquad [\hat{x}^\mu, \hat{p}^\mu] = \mathrm{i}\hbar\eta^{\mu\nu}\frac{\lambda^2}{L^2}\Sigma^{56} \to \mathrm{i}\hbar\eta^{\mu\nu}\mathbf{1}. \tag{3.27}$$

Rigorously speaking, when $\lambda \to 0$ the last commutator $[x^\mu, p^\nu] \to 0$ since the generator Σ^{56} is well defined. It is the large n limit of $\langle n|\Sigma^{56}|n\rangle$ that reproduces the ordinary Heisenberg uncertainty relations.

The dynamical consequences of Yang's noncommutative spacetime algebra can be derived from the quantum/classical correspondence

$$\frac{1}{\mathrm{i}\hbar}[\hat{A}, \hat{B}] \leftrightarrow \{A, B\}_{PB}, \tag{3.28}$$

i.e. commutators correspond to Poisson brackets. More precisely, to Moyal brackets in phase space. In the classical limit $\hbar \to 0$ Moyal brackets reduce to Poisson brackets. Since the coordinates and momenta are no longer commuting variables, the classical Newtonian dynamics is going to be modified since the symplectic 2-form $\omega^{\mu\nu}$ in phase space will have additional non-vanishing elements stemming from these noncommuting coordinates and momenta.

In particular, the modified brackets read now

$$\begin{aligned}\{\{A(x,p), B(x,p)\}\} &= \partial_\mu A\omega^{\mu\nu}\partial_\nu B = \{A(x,p), B(x,p)\}_{PB}\{x^\mu, p^\nu\}\\ &\quad + \frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial x^\nu}\{x^\mu, x^\nu\} + \frac{\partial A}{\partial p^\mu}\frac{\partial B}{\partial p^\nu}\{p^\mu, p^\nu\}.\end{aligned} \tag{3.29}$$

If the coordinates and momenta were commuting variables the modified bracket will reduce to the first term only:

$$\{\{A(x,p), B(x,p)\}\} = \{A(x,p), B(x,p)\}_{PB}\{x^\mu, p^\nu\} = \left[\frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial p^\nu} - \frac{\partial A}{\partial p^\mu}\frac{\partial B}{\partial x^\nu}\right]\eta^{\mu\nu}\mathcal{N}. \tag{3.30}$$

In the nonrelativistic limit, the modified dynamical equations are

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \{\{x^i, H\}\} = \frac{\partial H}{\partial p^j}\{x^i, p^j\} + \frac{\partial H}{\partial x^j}\{x^i, x^j\}. \tag{3.31}$$

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = \{\{p^i, H\}\} = -\frac{\partial H}{\partial x^j}\{x^i, p^j\} + \frac{\partial H}{\partial p^j}\{p^i, p^j\}. \tag{3.32}$$

The nonrelativistic Hamiltonian for a central potential $V(r)$ is

$$H = \frac{p_i p^i}{2m} + V(r) \qquad r = \left[\sum_i x_i x^i\right]^{1/2}. \tag{3.33}$$

Defining the magnitude of the central force by $F = -\frac{\partial V}{\partial r}$ and using $\frac{\partial r}{\partial x^i} = \frac{x_i}{r}$, one has the modified dynamical equations of motion

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \{\{x^i, H\}\} = \frac{p_j}{m}\delta^{ij} - F\frac{x_j}{r}L_P^2\Sigma^{ij}. \tag{3.34}$$

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = \{\{p^i, H\}\} = F\frac{x_j}{r}\delta^{ij} + \frac{p_j}{m}\frac{\Sigma^{ij}}{R^2}. \tag{3.35}$$

The angular momentum 2-vector Σ^{ij} can be written as the *dual* of a vector $\vec{J}$ as follows $\Sigma^{ij} = \epsilon^{ijk}J_k$ so that

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \{\{x^i, H\}\} = \frac{p^i}{m} - L_P^2 F\frac{x_j}{r}\epsilon^{ijk}J_k. \tag{3.36}$$

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = \{\{p^i, H\}\} = F\frac{x^i}{r} + \frac{p_j}{m}\frac{\epsilon^{ijk}J_k}{R^2}. \tag{3.37}$$

For planar motion (central forces) the cross-product of $\vec{J}$ with $\vec{p}$ and $\vec{x}$ is not zero since $\vec{J}$ points in the perpendicular direction to the plane. Thus, one will have nontrivial corrections to the ordinary Newtonian equations of motion induced from Yang's noncommutative spacetime algebra in the nonrelativistic limit. When $\vec{J} = 0$, pure radial motion, there are no corrections.

Concluding, equations (3.36)–(3.37) determine the *modified* Newtonian dynamics of a test particle under the influence of a central potential explicitly in terms of the two L_P, R minimal/maximal scales. When $L_P \to 0$ and $R \to \infty$ one recovers the ordinary Newtonian dynamics $v^i = (p^i/m)$ and $F(x^i/r) = m(\mathrm{d}v^i/\mathrm{d}t)$. The unit vector in the radial direction has for components $\hat{r} = (\vec{r}/r) = (x^1/r, x^2/r, x^3/r)$. The modified Newtonian dynamics represented by equations (3.36)–(3.37) should have important astrophysical consequences in the dynamics of the spiral arms of rotating galaxies at large distances from the centre.

4. Quantum mechanics in C-spaces

The most important result of this section is the emergence of a matrix-valued generalization of Planck's constant $\mathcal{H}^{AB}$ that is associated with a *novel* generalized Weyl–Heisenberg algebra in C-spaces. It is helpful, although not *necessary*, to derive the generalized Weyl–Heisenberg algebra associated with the quantum mechanics in C-spaces by recurring to matrix realizations of Clifford algebras. For example, vector coherent states have been defined on Clifford algebras, quaternions and octonions [49] that is the starting point to construct an (overcomplete) basis in a Hilbert space. This will be the subject of future investigations, in particular to study the noncommutative quantum oscillator in C-spaces by recurring to the poly-vector coherent states construction on Clifford algebras generalizing the work of [49]. Our aim in this section is more simple and we shall just focus on constructing the Weyl–Heisenberg algebras associated with the poly-coordinates and poly-momenta.

Due to the noncommutative nature of the basis vectors of the Clifford algebra one has

$$[E_A, E_B] = f_{AB}^M E_M = f_{ABM}E^M \qquad E_A = \{\mathbf{1}, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho, \ldots\}. \tag{4.1}$$

In order to raise and lower indices it requires the use of the C-space metric G^{AB} and G_{AB} respectively given by the scalar part of the geometric product of $\langle E^A E^B\rangle_{\text{scalar}} = \langle E^A E^B\rangle_0 = G^{AB}$, etc. The geometry of *curved* C-space involving G_{MN} was studied by [13] where it was shown how higher derivative gravity with torsion in ordinary spacetime emerged naturally from the scalar curvature in C-space. As mentioned above, we will choose to work with a finite value of D to avoid algebraic convergence problems.

The quantities f_{AB}^M play a similar role as the structure constants in ordinary Lie algebras. A commutator of two matrices is itself a matrix, which in turn, can be expanded in a suitable matrix basis due to the Clifford algebraic (vector space) structure inherent in C-spaces. The commutator algebra obeys the Jacobi identities, the Liebnitz rule of derivations and the antisymmetry properties.

The Clifford geometric product of two basis elements can be rewritten as

$$E_A E_B \equiv \frac{1}{2}\{E_A, E_B\} + \frac{1}{2}[E_A, E_B] = d^C_{AB} E_C + f^C_{AB} E_C. \tag{4.2}$$

In general the geometric product of two poly-vectors E^A, E^B of ranks r, s, respectively, is given by an aggregate of multivectors (poly-vectors) of the form

$$\begin{aligned} E^A E^B = \langle E^A E^B\rangle_{r+s}, \quad &\langle E^A E^B\rangle_{r+s-2}, \quad \langle E^A E^B\rangle_{r+s-4}, \ldots, \\ \langle E^A E^B\rangle_{r+s-6}, \ldots, \quad &\langle E^A E^B\rangle_{|r-s|}. \end{aligned} \tag{4.3}$$

The first term of rank $r+s$ is the wedge product $E^A \wedge E^B$ and the last term of rank $|r-s|$ is the dot product $E^A \bullet E^B$ which is obtained by a *contraction* of indices. In general, the scalar product among two equal-rank multivectors $r = s$ can no longer be written in terms of the anticommutator $\{E^A, E^B\}$ except in the case when $r = s = 1$, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1$. However, for equal-rank multivectors, the scalar part $\langle E_A E_B\rangle_0 = G_{AB}\mathbf{1}$, where $\mathbf{1}$ is the unit element of the Clifford algebra and G_{AB} is the C-space metric.

Our proposal for the generalized Heisenberg algebra in C-spaces is

$$\left[\frac{X^A}{L_P^{r_A}}, \left(\frac{L_P}{\hbar}\right)^{r_B} P^B\right] = \mathrm{i}\Theta^C \Omega^{AB}_C = \mathrm{i}\mathcal{H}^{AB} = \mathrm{i}G^{AB} + \text{EXTRA TERMS}, \tag{4.4}$$

where we have written the algebra in terms of dimensionless variables by means of explicitly introducing a length scale L_P (Planck's scale). r_A, r_B are the ranks of the poly-vectors X^A, P^B respectively. The C-space extension of Planck's constant is now matrix-valued $\mathcal{H}^{AB}$ and is encoded in the real-valued structure constants given by the scalar part of the triple product of the Clifford algebra generators

$$\Omega^{AB}_C \equiv \langle E_C E^A E^B\rangle_{\text{scalar}} = d^{AB}_C + f^{AB}_C. \tag{4.5a}$$

$$\frac{1}{2}\{E^A, E^B\} = d^{AB}_C E^C \qquad \frac{1}{2}[E^A, E^B] = f^{AB}_C E^C, \tag{4.5b}$$

with the additional commutators

$$[X^A, X^B] = 0 \qquad [P^A, P^B] = 0 \tag{4.6}$$

such that the algebra given by equations (4.4), (4.6) obeys the Jacobi identities. The Clifford-valued number $\Theta = \Theta^C E_C$, whose components Θ^C encode the matrix-valued $\Theta^C\Omega^{AB}_C = \mathcal{H}^{AB}$ extension of Planck's constant admits the expansion

$$\Theta = \Theta^C E_C = \theta\mathbf{1} + \theta^\mu\gamma_\mu + \theta^{[\mu\nu]}\gamma_\mu \wedge \gamma_\nu + \theta^{[\mu\nu\rho]}\gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho + \cdots. \tag{4.7}$$

In the following section, we will propose another generalized Heisenberg algebra in C-spaces that is associated with *noncommuting* coordinates and momenta that differ from the Clifford space Yang's algebra of [53].

Due to the fact that

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \eta^{\mu\nu}\mathbf{1} + \frac{1}{2}\gamma^\mu \wedge \gamma^\nu = \eta^{\mu\nu}\mathbf{1} + \frac{1}{2}\gamma^{[\mu\nu]}, \tag{4.8}$$

the triple products are

$$\langle \mathbf{1}\gamma^\mu\gamma^\nu\rangle_{\text{scalar}} = \eta^{\mu\nu} \tag{4.9a}$$

$$\langle \gamma_\rho \wedge \gamma_\tau\gamma^\mu\gamma^\nu\rangle_{\text{scalar}} = \langle \gamma_\rho \wedge \gamma_\tau\gamma^\mu \wedge \gamma^\nu\rangle_{\text{scalar}} = \delta^{\mu\nu}_{\rho\tau} = \delta^\mu_\rho\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\rho \tag{4.9b}$$

and the *modified* Weyl–Heisenberg algebra among coordinates and momenta is now

$$[x^\mu, p^\nu] = \mathrm{i}\hbar\theta\eta^{\mu\nu} + \mathrm{i}\hbar\theta^{[\mu\nu]}, \tag{4.10}$$

with profound consequences. One could set the parameter $\theta = 1$ in order to match the standard term of the Weyl–Heisenberg algebra. An immediate *new* consequence is that now one has the *modified* product of uncertainties given by

$$\Delta x \Delta p_x \geqslant \tfrac{1}{2}\theta\hbar, \qquad \Delta x \Delta p_y \geqslant \tfrac{1}{2}\hbar\theta^{xy}, \qquad \Delta x \Delta p_z \geqslant \tfrac{1}{2}\hbar\theta^{xz}, \ldots, \tag{4.11}$$

which imply (for example in $D = 4$) that one *cannot* simultaneously measure the *pairs* of quantities $(x, p_y), (x, p_z), (y, p_z) \ldots$ with absolute precision. This has to be compared with the standard QM product of uncertainties

$$\Delta x \Delta p_x \geqslant \tfrac{1}{2}\hbar, \qquad \Delta x \Delta p_y \geqslant 0, \qquad \Delta x \Delta p_z \geqslant 0, \ldots, \tag{4.12}$$

that permit a simultaneous measurement of $(x, p_y), (x, p_z), (y, p_z), \ldots$ with absolute precision!

Due to the geometric product of two bi-vector basis elements (omitting numerical factors)

$$\gamma^{\mu_1\mu_2}\gamma^{\nu_1\nu_2} = \gamma^{\mu_1\mu_2\nu_1\nu_2} + (\eta^{\mu_1\nu_2}\gamma^{\mu_2\nu_1} - \eta^{\mu_1\nu_1}\gamma^{\mu_2\nu_2} + \cdots) + \eta^{\mu_1\mu_2\nu_1\nu_2}\mathbf{1} \tag{4.13}$$

the bi-vectors commutators among area coordinates and area–momentum coordinates are

$$[x^{\mu_1\mu_2}, p^{\nu_1\nu_2}] = \mathrm{i}\hbar^2\Theta^C\Omega_C^{\mu_1\mu_2\nu_1\nu_2} = \mathrm{i}\hbar^2\Theta^C\langle E_C\gamma^{\mu_1\mu_2}\gamma^{\nu_1\nu_2}\rangle_{\text{scalar}}, \tag{4.14}$$

where the only non-vanishing quantities in the rhs are

$$\Theta^C\Omega_C^{[\mu_1\mu_2][\nu_1\nu_2]} = \theta\Omega_0^{[\mu_1\mu_2][\nu_1\nu_2]} + \theta^{[\rho_1\rho_2]}\Omega_{[\rho_1\rho_2]}^{[\mu_1\mu_2][\nu_1\nu_2]} + \theta^{[\rho_1\rho_2\rho_3\rho_4]}\Omega_{[\rho_1\rho_2\rho_3\rho_4]}^{[\mu_1\mu_2][\nu_1\nu_2]}. \tag{4.15}$$

The structure constants are obtained from the triple products

$$\langle\mathbf{1}\gamma^{\mu_1\mu_2}\gamma^{\nu_1\nu_2}\rangle_{\text{scalar}} = \Omega_0^{[\mu_1\mu_2][\nu_1\nu_2]} = \eta^{[\mu_1\mu_2][\nu_1\nu_2]} = \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1} \tag{4.16a}$$

$$\langle\gamma_{\rho_1\rho_2\rho_3\rho_4}\gamma^{\mu_1\mu_2}\gamma^{\nu_1\nu_2}\rangle_{\text{scalar}} = \Omega_{\rho_1\rho_2\rho_3\rho_4}^{\mu_1\mu_2\nu_1\nu_2} = \delta_{\rho_1\rho_2\rho_3\rho_4}^{\mu_1\mu_2\nu_1\nu_2} \tag{4.16b}$$

$$\langle\gamma_{\rho_1\rho_2}\gamma^{\mu_1\mu_2}\gamma^{\nu_1\nu_2}\rangle_{\text{scalar}} = \Omega_{\rho_1\rho_2}^{\mu_1\mu_2\nu_1\nu_2} = \eta^{\mu_1\nu_2}\delta_{\rho_1\rho_2}^{\mu_2\nu_1} - \eta^{\mu_1\nu_1}\delta_{\rho_1\rho_2}^{\mu_2\nu_2} + \cdots. \tag{4.16c}$$

Therefore from equations (4.14)–(4.16), one learns (after setting $\theta = 1$) that

$$\Delta x^{\mu_1\mu_2}\Delta p^{\mu_1\mu_2} \geqslant \hbar^2, \text{(no index summation)}\, \Delta x^{\mu_1\mu_2}\Delta p^{\nu_1\nu_2} \neq 0, \quad \text{when } \mu_1 \neq \nu_1, \mu_2 \neq \nu_2. \tag{4.16d}$$

In the derivation of equation (4.16a) we used the result that a *flat* C-space metric, for example, has for components

$$G^{\mu\nu} = \eta^{\mu\nu} \qquad G^{[\mu_1\mu_2][\nu_1\nu_2]} = \eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2} - \eta^{\mu_1\nu_2}\eta^{\mu_2\nu_1}, \ldots. \tag{4.17a}$$

It is convenient to order the indices in an increasing sequence:

$$\mu_1 < \mu_2 < \mu_3 \cdots < \mu_n \qquad \nu_1 < \nu_2 < \nu_3 \cdots < \nu_n. \tag{4.17b}$$

The Planck scale L_P is going to *explicitly* appear in the commutators of poly-coordinates and poly-momenta of *different* rank. For example, in commutators like

$$\left[\frac{x^\mu}{L_p}, \left(\frac{L_P}{\hbar}\right)^2 p^{\nu_1\nu_2}\right] = \mathrm{i}\theta^\rho\Omega_\rho^{\mu\nu_1\nu_2} + \mathrm{i}\theta^{\rho_1\rho_2\rho_3}\Omega_{\rho_1\rho_2\rho_3}^{\mu\nu_1\nu_2}, \tag{4.18}$$

where the structure constants are

$$\Omega_{\rho_1\rho_2\rho_3}^{\mu\nu_1\nu_2} = \langle\gamma_{\rho_1\rho_2\rho_3}\gamma^\mu\gamma^{\nu_1\nu_2}\rangle_{\text{scalar}} = \delta_{\rho_1\rho_2\rho_3}^{\mu\nu_1\nu_2} \tag{4.19a}$$

$$\Omega_\rho^{\mu\nu_1\nu_2} = \langle\gamma_\rho\gamma^\mu\gamma^{\nu_1\nu_2}\rangle_{\text{scalar}} = \eta^{\mu\nu_1}\delta_\rho^{\nu_2} - \eta^{\mu\nu_2}\delta_\rho^{\nu_1} \tag{4.19b}$$

resulting from the geometric product of a vector with a bi-vector basis element which can be decomposed, up to numerical factors, into a sum of a 3-vector and a vector as

$$\gamma^{\mu}\gamma^{\nu_1\nu_2} = \gamma^{\mu\nu_1\nu_2} + \eta^{\mu\nu_1}\gamma^{\nu_2} - \eta^{\mu\nu_2}\gamma^{\nu_1}. \tag{4.20}$$

As mentioned above, the most important conclusion was obtained from equation (4.11) and is that now one cannot simultaneously measure the pairs of quantities $(x, p_y), (x, p_z), (y, p_z), \ldots$ with absolute precision. Similar products of uncertainties apply to the remaining commutators involving poly-coordinates and poly-momenta of different and/or equal ranks.

The second most important result is that *novel* QM wave equations in C-space can be found compatible with the novel Weyl–Heisenberg algebra. These novel wave equations contain, for example, the Dirac–Lanczos quaternionic wave equation which was later rediscovered by Barut–Hestenes. These novel wave equations are obtained by recurring to the realization of the poly-momentum operator in natural units of $\hbar = c = L_p = 1$ such that is consistent with the Weyl–Heisenberg algebra as follows:

$$\begin{aligned} &P^B \rightarrow \mathrm{i}\Theta^C\Omega_C^{DB}\frac{\partial}{\partial X^D} = \mathrm{i}\mathcal{H}^{DB}\frac{\partial}{\partial X^D} \Rightarrow \\ &[X^A, P^B] = \left[X^A, \mathrm{i}\Theta^C\Omega_C^{DB}\frac{\partial}{\partial X^D}\right] = \mathrm{i}\Theta^C\Omega_C^{DB}\delta_D^A = \mathrm{i}\Theta^C\Omega_C^{AB} = \mathrm{i}\mathcal{H}^{AB}. \end{aligned} \tag{4.21}$$

Due to this new realization of the poly-momentum operator it will lead to novel QM wave equations in C-space and novel dispersion relations. For example, the novel version of the Klein–Gordon and Dirac wave equations in C-spaces are respectively

$$G_{AB}\left[\mathrm{i}\Theta^C\Omega_C^{DA}\frac{\partial}{\partial X^D}\right]\left[\mathrm{i}\Theta^C\Omega_C^{DB}\frac{\partial}{\partial X^D}\right]\Phi(X) = \mathcal{M}^2\Phi(X). \tag{4.22a}$$

$$\left[\mathrm{i}E_B\Theta^C\Omega_C^{DB}\frac{\partial}{\partial X^D}\right]\Psi(X) = \mathcal{M}\Psi(X). \tag{4.22b}$$

Note that when one *truncates* the components of the Θ^C parameters to zero, except the first one $\theta \neq 0$ and which we set to unity, equation (4.22*b*) becomes, in the natural units of $\hbar = L_{\text{Planck}} = 1$:

$$-\mathrm{i}\left(\frac{\partial}{\partial\Omega} + \gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + \gamma^{\mu}\wedge\gamma^{\nu}\frac{\partial}{\partial x^{\mu\nu}} + \cdots\right)\Psi(\Omega, x^{\mu}, x^{\mu\nu}, \ldots) = M\Psi(\Omega, x^{\mu}, x^{\mu\nu}, \ldots). \tag{4.23}$$

Ordinary spinors are nothing but elements of the left/right ideals of a Clifford algebra. So they are automatically contained in the poly-vector-valued wavefunction Ψ.

Note that the approach based on equation (4.23) is different from that by Hestenes who proposed an equation which is known as the Dirac–Hestenes equation. Dirac's equation using quaternions (related to Clifford algebras) was first derived by Lanczos [60]. Later on the Dirac–Lanczos equation was rediscovered by many people, in particular by Hestenes and Gursey in what became known as the Dirac–Hestenes equation. The former Dirac–Lanczos equation is Lorentz *covariant* despite the fact that it singles out an arbitrary but unique direction in ordinary space: the *spin* quantization axis. Lanczos, without knowing, had anticipated the existence of isospin as well. The Dirac–Hestenes equation

$$(\gamma^{\mu}\partial_{\mu}\Psi)\gamma_2\wedge\gamma_1 = m\Psi\gamma_0 \tag{4.24}$$

is *covariant* under a change of frame $\gamma'_{\mu} = U\gamma_{\mu}U^{-1}$ and $\Psi' = \Psi U^{-1}$ with U an element of the $\text{Spin}_+(1, 3)$ yielding $\partial\Psi' e'_{21} = m\Psi' e'_0$. As Lanczos had anticipated, in a new frame of reference, the spin quantization axis is also rotated appropriately, thus there is no breakdown of covariance by introducing bi-vectors in the Dirac–Hestenes equation.

However, subtleties still remain. In the Dirac–Hestenes equation instead of the imaginary unit i there occurs the bi-vector $\gamma_1 \wedge \gamma_2$. Its square is -1 and commutes with all the elements of the Dirac algebra which is just a desired property. But on the other hand, the introduction of a bi-vector into an equation implies a selection of a preferred orientation in spacetime, i.e. the choice of the spin quantization axis in the original Dirac–Lanczos quaternionic equation. How is such preferred orientation (spin quantization axis) determined? Is there some dynamical symmetry which determines the preferred orientation (spin quantization axis)? Is there an action which encodes a hidden dynamical principle that selects *dynamically* a preferred spacetime orientation (spin quantization axis)? A monograph devoted to quaternionic QM can be found in [59].

In the most general case when the 2^D components of Θ^C are not zero, one can see that equation (4.22b) *contains* more terms than equation (4.23). One can diagonalize the matrix-valued Planck constant $\mathcal{H}^{AB}$, that has $2^D \times 2^D$ components, into a diagonal matrix with 2^D components which matches precisely the 2^D components present in the Θ^C parameter. Thus, roughly speaking, the novel Dirac-equation (4.22b) contains 2^D copies of equation (4.23) if one were to diagonalize the $2^D \times 2^D$ matrix-valued Planck constant $\mathcal{H}^{AB}$ into a diagonal matrix with 2^D entries. Equations (4.22b), (4.23) are more general than the equations studied in [1] involving quaternions and complex quaternions to account for bispinor wavefunctions. A generalization of Yang–Mills theories based on *tensorial* gauge field theories in C-spaces and higher spins extensions of the standard model was studied in [54] along with generalized poly-vector-valued supersymmetries in C-spaces.

The third important conclusion of this section is that a *novel* matrix-valued dispersion relation between the poly-momentum and the poly-wave-vector can be obtained from the novel Klein–Gordon equation, associated with plane wave solutions, and is given by $P^A = \mathcal{H}^{AB} K^B$ which is the generalization of the de Broglie relation $p^\mu = \hbar k^\mu$.

5. Concluding remarks

We conclude with a brief discussion on other algebras, besides the Generalized Yang's noncommutative algebra in C-spaces [53], involving *noncommuting* poly-coordinates and poly-momenta. To simplify matters by not having to keep track of the units we will choose the natural units $\hbar = c = L_P = 1$. The standard noncommutative algebra in C-spaces must be of the form

$$[X^A, X^B] = \mathrm{i}\Sigma^{AB} \qquad [P^A, P^B] = \mathrm{i}\Theta^{AB} \qquad [X^A, P^B] = \mathrm{i}G^{AB} + \frac{\mathrm{i}}{4}\Sigma^{AM}\Theta^{MB}, \tag{5.1}$$

where G^{AB} is *flat* C-space metric and the structure constants (c-numbers) $\Sigma^{AB} = -\Sigma^{BA}$ and $\Theta^{AB} = -\Theta^{BA}$ obey the conditions

$$[X^A, \Sigma^{BC}] = [P^A, \Sigma^{BC}] = [X^A, \Theta^{BC}] = [P^A, \Theta^{BC}] = [X^A, G^{BC}] = [P^A, G^{BC}] = 0. \tag{5.2}$$

A non-canonical change of coordinates

$$X'^A = X^A + \tfrac{1}{2}\Sigma^{AC}P^C \qquad P'^A = P^A + \tfrac{1}{2}\Theta^{AC}X^C. \tag{5.3}$$

leads to an algebra with commuting coordinates and momenta but with a modified $[X, P]$ commutator

$$[X'^A, X'^B] = 0 \qquad [P'^A, P'^B] = 0$$
$$[X'^A, P'^C] = \mathrm{i}G^{AC} + \frac{\mathrm{i}}{2}\Sigma^{AM}\Theta^{MC} + \frac{\mathrm{i}}{16}\Sigma^{AB}\Theta^{BN}\Sigma^{ND}\Theta^{DC}. \tag{5.4}$$

A more general algebra involving noncommuting coordinates/momenta, and related to the minimal length stringy-uncertainty relations of section 2.2, can be constructed by implementing the idea of 2.2 behind an *effective* Planck constant which is energy–momentum dependent $\hbar_{\text{eff}}(p^2)$, i.e. the $[X, P]$ commutator must involve quadratic terms in the poly-momenta $P^2 = P^A P_A$, which in turn, can be expanded into a sum of powers of the ordinary momentum $p^\mu p_\mu$, as explained in 2.2.

Concluding, the results of sections 2.2, 3.1, 3.2 and 4 are all very different. The noncommutative algebra given by commutation relations of equations (5.1) is different than Yang's algebra of section 3 and the novel Weyl–Heisenberg algebra of section 4 based on a matrix-valued $\mathcal{H}^{AB}$ extension of Planck's constant. The latter algebra permits to construct the *novel* QM wave equations in C-spaces with profound new physical consequences, i.e. there are modified products of uncertainties such that one cannot longer measure simultaneously the pairs of variables (x, p_x), (x, p_y), (x, p_z), . . . and there are also modified dispersion relations in contradistinction to what occurs in ordinary QM.

We have explained in the introduction and in section 2 why the Planck scale is a true invariant of C-space that is required in order to combine p-loops (closed p-branes) of different dimensions. C-space relativity contains two fundamental constants, the speed of light and the Planck scale. The authors in [11, 12, 21] have interpreted the Planck scale as a *deformation* parameter in the *kappa*-deformed Poincare algebras $l = 1/\kappa = \lambda$ (used in double special relativity theories), where the ordinary four-dimensional Lorentz invariance is *broken* explicitly and only the rotational symmetry is preserved.

Quantum group deformations of the Poincare and conformal group have been developed by Castellani [20] and used in his construction of a bicovariant q-gravity theory. One may interpret the q-deformation parameter in terms of a minimal Planck scale and an upper impassible scale R by setting $q = \exp(\lambda/R)$ so that when λ goes to zero, or when R goes to infinity, the deformation parameter collapses to the classical undeformed values $q = 1$. Hence, the classical gravitational theory is recovered in the short and large distance limits of Castellani's q-gravity theory. This sort of ultraviolet/infrared entanglement duality has received a lot of attention in the past years within the framework of noncommutative QFT defined on noncommutative spacetimes and in M-theory [19, 43].

We have argued in [37] why the *kappa*-deformed Poincare algebras could be obtained directly from an 8D phase space via a Moyal star product deformation procedure by taking the Planck scale as the deformation parameter $\kappa = 1/m_P = L_P$ and by choosing an appropriate basis $X^\mu(x, p, \kappa)$, $P^\mu(x, p, \kappa)$ of the 8D phase-space coordinates such that the Moyal bracket algebra involving the Lorentz and translation generators associated with the new X^μ, P^μ coordinates is isomorphic to the *kappa*-deformed Poincare algebra in terms of the old x, p coordinates.

It is warranted to explore the relationship among all these algebras on a unified footing. Two-parameter quantum Hopf algebraic deformations of ordinary algebras have been studied in the past in the context of quantum groups [20] and more recently by the authors [38, 42]. Unfortunately, the latter authors were not aware of the old work by Yang [17] on noncommutative spacetime algebras which involved *two* different length scales and of Tanaka's work [18] about the connection of Yang's algebra to de Sitter and Anti de Sitter spaces and the physical explanation of the origins of a *discrete* spectrum for the spatial coordinates and spatial momenta that yield a *minimum* length λ (ultraviolet cutoff in energy) and a minimum momentum $p = \hbar/R$ (maximal length R, infrared cutoff). The importance of Yang's algebra and the Lie-algebraic stability in the construction of physical theories within the context of new length scales was addressed by [51, 52].

An upper limiting scale in cosmology was long ago advocated by Nottale [1] in his proposal for the resolution of the cosmological constant problem. It is unknown at the present if Nottale's fractal spacetime construction belongs to the class of noncommutative geometries studied by Connes. The importance of nonlinear dynamics, chaos and fractals in particle physics and cosmology has been raised by Nottale and others. Smith [50] has derived the values of all the coupling constants and masses of the standard model based on Clifford algebraic methods associated with hyper-diamond discrete lattices (a Feyman chessboard model) generalizing the celebrated Wyler's mathematical expression for the fine structure constant. Most recently, Beck [39] gave convincing numerical results to support why chaotic strings dynamics determines the values of all the standard model parameters.

Despite the fact that Clifford algebras could be interpreted already as the quantum extensions of Grassmann algebras we believe that a main task in the near future will be to construct QFT's in C-spaces based on quantum Clifford algebras, like the braided Hopf quantum Clifford algebras [15]. It has been argued by Ablamowicz and Fauser [15] that these Hopf algebraic structures will replace groups and group representations as the leading paradigm in forthcoming times and why the Grassmann–Cayley bracket algebras and other algebraic structures are all covered by the Hopf algebraic framework. Recently, a Dirac–Kahler fermion action based on a new Clifford product with a noncommutative differential form on a lattice was introduced in [40].

A Moyal-like star product construction in C-spaces deserves further study as well. C-space involves the physics of all p-loops (closed p-branes); thus it is warranted to use methods of multisymplectic geometry (mechanics) due to the presence of antisymmetric tensors of arbitrary rank. Nambu–Poisson QM seems to be the most appropriate one to study C-space QM. In particular the use of the Zariski and Fedosov star product deformations versus the Moyal one [24] will be welcome. A *relativistic* variant of the Moyal–Wigner function was proposed by [47].

To conclude this work quantization in C-spaces contains a very rich noncommutative structure from which many old results can be derived after breaking the C-space Lorentz covariance/invariance (pan-dimensional covariance). No extra dimensions are required to introduce a length scale. C-space relativity already has a natural invariant minimum Planck scale by definition. The Weyl–Heisenberg algebra in C-spaces is naturally modified due to the noncommutativity of the Clifford algebra basis elements. In essence, when both QM and relativity are extended to C-spaces by means of introducing a matrix-valued Planck constant and poly-vectors both QM and relativity theories are modified accordingly which maybe what is required in order to formulate a consistent quantum theory of gravity.

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